Analytical Methods in Component Modal Synthesis

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A technique for modeling a structure using a severely truncated mode set is developed. Implications of transforming finite-element model equations of motion into a new generalized coordinate space, and truncating the new degrees of freedom are examined. The equations of constraint introduced by such truncation are explicitly stated for several coordinate spaces. A basis is established for alternately viewing modal truncation as a truncation of inertia relief modes, a Guyan reduction, or a truncation of normal modes. It is established that the Hurty component mode set may be viewed as equivalent to a Guyan reduction of a component structure to its interface degrees of freedom supplemented with a cantilever normal mode set. It is suggested that an interface mode set selection be based upon a static analysis of the component structure response due to imposed interface forces and displacements. The technique yields benefits similar to modal acceleration in forced response analysis, and results in the inclusion of inertia relief response functions in the interface mode set. Freedom from cantilever boundary conditions and other types of interface constraints on a component normal mode set results. Improved convergence in synthesis is demonstrated in a numerical example. The interface mode set may be used with additional degrees of freedom to perform static analysis of the synthesized system.

Nomenclature

$\bar{B},\bar{C},\bar{D},\bar{E}$, $\bar{F},\bar{G},\bar{I},\bar{L}$, $\bar{P},\bar{R},\bar{U},\bar{V},\bar{W}$ = physical coordinate sets in finite element model
$\bar{b},\bar{c},\bar{d},\bar{e}$, $\bar{f},\bar{g},\bar{i},\bar{\ell}$, $\bar{p},\bar{r},\bar{u},\bar{v},\bar{w}$ = number of coordinates in $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, $\bar{F}$, $\bar{G}$, $\bar{I}$, $\bar{L}$, $\bar{P}$, $\bar{R}$, $\bar{U}$, $\bar{V}$, $\bar{W}$, respectively
F = force matrix or vector
I = identity matrix
k = stiffness matrix
m = mass matrix
M = inertia relief loading matrix, Eq. (8)
p_C = component generalized displacement vector
q^C = constraint mode generalized displacement vector
R = interface reaction force vector
S = coordinate transformation matrix, Eq. (18)
x = physical coordinate vector
y = vector of independent component modal degrees of freedom in a coupled system, Eq. (24)
β = coupling transformation matrix, Eq. (25)
$\mathfrak{M}$ = generalized mass matrix $\phi^T m \phi$
κ = generalized stiffness matrix $\phi^T k \phi$
ϕ = mode shape matrix
ϕ^a = attachment mode shape matrix
ϕ^C = constraint mode shape matrix
ϕ^n = normal mode shape matrix
ϕ^r = rigid body mode shape matrix
ϕ^s = system normal mode shape matrix, Eq. (31)

Subscripts

i,j = component i,j, respectively
b,c,d, f,ℓ,v,w = matrix partitions in $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{F}$, $\bar{L}$, $\bar{V}$, $\bar{W}$ sets, respectively
λ = retained set of normal modes
ρ = truncated set of normal modes

Superscripts

T = matrix transpose
c,f = matrix partitions defined in Eq. (14)

Received May 17, 1974; revision received January 20, 1975.
Index categories: Structural Dynamic Analysis; Structural Static Analysis.
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I. Introduction

THE successful design of structures requires analysis for internal loads when the structure is placed in its operating environment. A vital part of this effort is the modal analysis of structural finite-element models. In the classical approach, it is usual to determine normal modes and auxiliary statics analysis directly from the finite-element model. Complete structural systems have become very complex and major components are often produced by different organizations. Therefore, it is often difficult to assemble an entire finite-element model in a timely manner. In addition, many finite-element models may contain so many degrees of freedom that they cannot be handled directly on the largest of modern computers. For these reasons, it is desirable to develop methods for analyzing substructures of a finite-element model. Such analysis has come to be known as "component modal syntheses" in dynamics and "substructure analysis" in statics. The methods of this paper are formulated for dynamic analysis. However, there is a great deal of application in statics substructure analysis. It is desirable that component mode techniques for dynamic analysis of structures have the following characteristics.

1) *Computational efficiency*: The component mode representation should contain a minimum number of independent degrees of freedom or modes for each component.

2) *Interchangeability*: The component mode set should be independent of the inertial and stiffness properties of adjacent components. Such a component mode set may be used interchangeably in different structural systems with compatible interfaces.

3) *Component flexibility*: Methods should permit optional interface degrees of freedom in a component mode set that may be used or discarded at the time of syntheseis. Such a mode set need not be redefined for each potential interface or potential combination of interfaces.

4) *Synthesis flexibility*: A synthesis technique should not be constrained to a particular type of component mode set. Synthesis techniques should be amenable to accepting different types of component mode sets (i.e., fixed interface, free interface, inertia loading, etc.).

5) *Modal acceleration analysis*: Static solutions for the synthesized system, which may be required for modal ac-

Table 1 Definition of physical coordinate sets in a structural model

Set	Size	Description
$\bar{P}$	$\bar{p}$	All physical degrees of freedom in model
$\bar{I}$	$\bar{i}$	Inertial degrees of freedom in model which include at least one $\bar{R}$ set ($\bar{p}\geq\bar{i}\geq\bar{r}$)
$\bar{R}$	$\bar{r}$	Any arbitrarily chosen, nonredundant set of constraints sufficient to remove any existing rigid body degrees of freedom ($\bar{p}>\bar{r}\geq o$)
$\bar{U}$	$\bar{u}=\bar{p}-\bar{r}$	Complement of $\bar{R}$ set in $\bar{P}$
$\bar{B}$	$\bar{b}$	Any subset of $\bar{P}$ including at least one $\bar{R}$ set ($\bar{p}\geq\bar{b}\geq\bar{r}$)
$\bar{V}$	$\bar{v}=\bar{p}-\bar{b}$	Complement of $\bar{B}$ set in $\bar{P}$
$\bar{L}$	$\bar{\ell}$	Any subset of $\bar{U}$($\bar{u}\geq\bar{\ell}\geq 1$)
$\bar{W}$	$\bar{w}=\bar{u}-\bar{\ell}$	Complement of $\bar{L}$ in $\bar{U}$
$\bar{G}$	$\bar{g}$	Degrees of freedom in $\bar{R}$ set without inertial properties

celeration forced response analysis, should be available. Such static analysis should be available, regardless of the complexity of the component interfaces, without requiring the assembly of a system finite element model.

It will be shown that the methods of this paper exhibit all of the preceding characteristics. Previous work in the literature has provided the basis for a general theory of component modal synthesis, as well as static substructure analysis with modes. Constraint modes have been defined by Hurty[1] and Craig and Bampton.[2] Attachment modes have been defined by Bamford[3] for a restrained model. Rubin[4] has extended the Bamford definition to the unrestrained model resulting in the definition of inertia relief modes. Goldenberg and Shapiro[5] have defined a general synthesis coupling technique. Their technique provides the basis for using any set of linearly independent modes in synthesis.

The selection of normal modes in a structural dynamic analysis is simply a matter of numerical convenience which minimizes computational effort. The Guyan reduction[6] is an example of the implicit use of modes in the solution of statics problems. For a particular kind of analysis there will generally be a small subset of the total number of degrees of freedom in a finite-element model which are significantly "active." All other degrees of freedom may be assumed to yield no response with no approximation to the finite-element model for static analysis and only a slight loss of accuracy in dynamic analysis. The science of selecting active degrees of freedom is to a large extent the subject of this paper.

Mode synthesis implicity includes the use of displacement-dependent interface reaction forces. Synthesis and forced response analysis with displacement-dependent external forces therefore require a precise determination of structure displacement response at interfaces. Truncation errors in these analyses will yield inaccuracies comparable to those in modal displacement analysis unless special care is taken.† As in modal displacement, these truncation errors are directly related to the incomplete static displacement representation of a finite-element model by a truncated normal mode set.

In this paper, static constraint and attachment modes are used to formulate interface mode sets. The inertia relief mode is included in the class of static modes and forms a subset of the class of modes called attachment modes. The attachment mode is defined as a static response mode resulting from imposed constant forces. The constraint mode is defined[2] as a static response mode resulting from imposed constant displacements.

Two interface mode sets are defined in this paper as the "method of attachment modes" and the "method of constraint modes." Each set includes both attachment modes and constraint (or rigid body) modes. Either set will yield precisely the same results as the original finite-element model solution to statically imposed, constant interface forces and displacements. Each set is therefore said to be statically complete. These interface mode sets may be used with the synthesis techniques of this paper for both dynamic and static analysis of structures. The interface mode set is supplemented with component normal modes for dynamic analysis. It is supplemented with additional static modes for static substructure analysis.

A sample problem is presented which compares methods of this paper to Hurty's method for an unrestrained two-component synthesis with a redundant interface. The basis of comparison is frequency error in synthesized eigenvalues for several different component mode truncations. Substantial improvement in frequency error is demonstrated. In this example, improvement is from very small frequency error to very, very small frequency error and may therefore appear to be of academic interest only. However, in other cases improvement may be of great practical value. This is especially so in the computations of stress which are much more sensitive to truncation than frequency error.

†In previous synthesis analyses, this special care has been typified by the cantilever interface constraints on component normal modes by Hurty and inertia and/or elastic interface loading by others.

II. Basis for Method

An insight into the meaning of truncation of mode sets in structural analysis is desired. The approach taken is based upon transformation theory and set concepts of linear algebra. The subject of concern is this: once equations of motion are formulated, what are the implications of transforming said equations into various new generalized coordinate spaces and truncating the new degrees of freedom?

Four mode classes are defined for this analysis: normal modes, rigid body modes, constraint modes, and attachment modes. They are employed to formulate five untruncated mode sets for an unrestrained finite-element model with inertial properties at all physical degrees of freedom. Each untruncated mode set is a linearly independent, reversible transformation resulting in a complete generalized coordinate space. Analysis in any of the five spaces will yield precisely the same results as the original finite-element model. Implications of physical degrees of freedom without inertial properties and truncation of the mode sets are considered for each space. Definitions required for this analysis and subsequent syntheses analysis are listed in Table 1 and Sec. IIA.

A. Definition of Mode Classes

1) *Rigid body modes*: Mode set of $\bar{r}$ modes which defines rigid body response of structure to unit displacements on $\bar{R}$ set.

2) *Normal modes:* Elastic mode set resulting in a diagonal generalized mass and stiffness matrix.

3) *Constraint modes:* These modes are defined by unit displacements imposed on any $\bar{B}$ set. One constraint mode is defined by imposing a unit displacement on one coordinate in a $\bar{B}$ set and zero displacement on all remaining coordinates in that $\bar{B}$ set. The constraint mode is the resulting response of all coordinates in the $\bar{P}$ set. There are $\bar{b}$ such linearly independent constraint modes for any $\bar{B}$ set. There are reaction forces F_b on the B set when $\bar{b}>\bar{r}$. The F_b matrix contains $\bar{r}$ singularities. When $\bar{b}=\bar{r}>0$ reaction forces are zero and rigid body modes result. Constraint modes are determined from Eq. (1) in partition form.[2]

$$\begin{bmatrix} k_{bb} & k_{bv} \\ k_{vb} & k_{vv} \end{bmatrix} \begin{bmatrix} I \\ \phi_v^C \end{bmatrix} = \begin{bmatrix} F_b \\ 0 \end{bmatrix} \tag{1a}$$

$$\phi^C = \begin{bmatrix} I \\ \phi_v^C \end{bmatrix} = \begin{bmatrix} I \\ -k_{vv}^{-1} k_{vb} \end{bmatrix} \tag{1b}$$

4) *Attachment modes*: These modes are defined by unit forces imposed on any $\bar{L}$ set. One attachment mode is defined by imposing a unit force on one coordinate in an $\bar{L}$ set and zero force on all remaining coordinates in that $\bar{L}$ set. The attachment mode is the response of all coordinates in the $\bar{P}$ set. There are $\bar{\ell}$ such linearly independent attachment modes for any $\bar{L}$ set. If the structure is unrestrained this mode set will consist of inertia relief modes. All such modes will be called static modes in this paper as the elastic structure is in equilibrium, albeit the unrestrained structure is equilibrated by D'Alembert inertia forces.

For a restrained structure, attachment modes are defined by Eq. (2) in parition form.[3]

$$\begin{bmatrix} k_{\ell\ell} & k_{\ell w} \\ k_{w\ell} & k_{ww} \end{bmatrix} \begin{bmatrix} \phi^a_\ell \\ \phi^a_w \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{2a}$$

$$\phi^a = \begin{bmatrix} \phi^a_\ell \\ \phi^a_w \end{bmatrix} \tag{2b}$$

Note that for this case $\bar{R}=0$ and the $\bar{U}$ set becomes equal to the $\bar{P}$ set. If attachment modes are defined for the same degrees of freedom as the constraint modes, then $\ell=b$ and $w=v$, and Eq. (2) becomes

$$\phi^a = \begin{bmatrix} \phi^a_b \\ \phi^a_v \end{bmatrix} = \begin{bmatrix} I \\ -k^{-1}_{vv}\, k_{vb} \end{bmatrix} \phi^a_b \tag{3}$$

The attachment modes of Eq. (2) are thus a linear combination of the constraint modes of Eq. (1) for this special case. The transformation relating the two sets is

$$\phi^a = \phi^C \phi^a_b \tag{4}$$

For an unrestrained structure, attachment modes are defined by the response of a uniformly accelerating system.[4] The equation of motion is given by

$$m\ddot{x} + kx = F \tag{5}$$

Since the structure is undergoing only rigid body accelerations

$$\ddot{x} = \phi^r[(\phi^r)^T m\phi^r]^{-1}(\phi^r)^T F \tag{6}$$

where ϕ^r is the structure rigid body mode shape matrix.

Substituting Eq. (6) into Eq. (5) yields

$$kx = F - m\phi^r[(\phi^r)^T m\phi^r]^{-1}(\phi^r)^T F \tag{7}$$

Any set of nonredundant constraints sufficient to put the structure in equilibrium may be imposed on k to solve the singular system of Eq. (7). In so doing, the constrained coordinates will have an imposed displacement of zero. The constraint reaction forces, F_r will also be zero since all external forces are equilibrated by the internal inertia relief forces. Let

$$M = I - m\phi^r[(\phi^r)^T m\phi^r]^{-1}(\phi^r)^T \tag{8}$$

Substituting Eq. (8) into Eq. (7), and defining attachment modes for the unrestrained structure in partition form yields

$$\begin{bmatrix} k_{\ell\ell} & k_{\ell w} & k_{\ell r} \\ k_{w\ell} & k_{ww} & k_{wr} \\ k_{r\ell} & k_{rw} & k_{rr} \end{bmatrix} \begin{bmatrix} \phi^a_\ell \\ \phi^a_w \\ 0 \end{bmatrix} = \begin{bmatrix} M_{\ell\ell} & M_{\ell w} & M_{\ell r} \\ M_{w\ell} & M_{ww} & M_{wr} \\ M_{r\ell} & M_{rw} & M_{rr} \end{bmatrix} \begin{bmatrix} I \\ 0 \\ F_r \end{bmatrix} \tag{9a}$$

$$\phi^a = \begin{bmatrix} \phi^a_\ell \\ \phi^a_w \\ 0 \end{bmatrix} \tag{9b}$$

B. Analysis of an Unrestrained Structure

For the moment consider the case of nonzero inertial properties in all physical degrees of freedom such that $\bar{p}=\bar{i}$. Some of the possible independent mode sets which are untruncated representations of the unrestrained finite element model are: 1) $\bar{i}-\bar{r}$ unrestrained normal modes plus $\bar{r}$ rigid body modes; 2) $\bar{i}-\bar{r}$ normal modes cantilevered at any $\bar{R}$ set plus $\bar{r}$ rigid body modes (a subset of set 3, with $\bar{b}=\bar{r}$); 3) $\bar{b}-\bar{b}$ normal modes cantilevered at any $\bar{B}$ set plus $\bar{b}$ constraint modes for that $\bar{B}$ set; 4) $\bar{p}-\bar{r}$ attachment modes defined for $\bar{L}=\bar{U}$ plus $\bar{r}$ rigid body modes; and 5) $\bar{p}$ constraint modes defined for $\bar{B}=\bar{P}$. In this configuration the mode shape matrix is an identity matrix.

All five of these mode set descriptions result in $\bar{p}\times\bar{p}$ linearly independent mode shape matrices. As such, any of the previous sets yields a transformed set of equation of motion which is complete. This is to say the structure may be analyzed with any of these equation sets and the resulting solutions in physical coordinates will agree precisely with solutions from the original finite-element model.

Now consider the implications of the general case in which physical coordinates exist without inertial properties ($\bar{p}>\bar{i}$) for the five mode sets previously listed: 1) This mode set will now represent a reduction of the original finite-element model from $\bar{p}$ elastic degrees of freedom to $\bar{i}$ elastic degrees of freedom. 2) This mode set will now represent a reduction of the original finite element model from $\bar{p}$ elastic degrees of freedom to $\bar{i}+\bar{g}$ elastic degrees of freedom. 3) No reduction in $\bar{B}$ set. All other physical degrees of freedom ($\bar{V}$ set) without inertia are transformed into dependent degrees of freedom. 4) No reduction if all coordinates without inertial properties are in $\bar{L}$ set. 5) No reduction.

Mode sets 1, 2, and 3 no longer yield an elastic representation equivalent to the finite element model. Degrees of freedom without inertia in the normal mode sets have no forces applied to them. These degrees of freedom have become dependent degrees of freedom, hence deflection patterns may only be determined to an approximation if external forces are applied to any of these degrees of freedom. In all other respects the normal mode sets will retain the detail of the original finite element model.

Implications of truncating degrees of freedom, *other than rigid body degrees of freedom*, are now considered. It is fundamental that the truncation of any independent degree of freedom from a finite-element model imposes an equation of constraint upon the model. For the truncation considered in mode sets 1-5, all equations of constraint may be viewed in terms of the forces on the model. The simplest equation of constraint is for static analysis wherein a set of external forces may be set to zero as in the Guyan reduction of Ref. 6. The resulting mode set may be used for static analysis with no approximation for selected forces and is thus statically complete with respect to those forces.

Truncation of mode set 4 results in $\bar{L}$ becoming a subset of $\bar{U}$ and the equations of constraint $F_w=0$. The retained set of modes contains the detail of the original finite element model with respect to static uniform acceleration response to forces at all $\bar{L}$ set degrees of freedom. Gross errors may be generated by imposing forces, or displacements which result in constraint forces, on any $\bar{W}$ set degrees of freedom.

Truncation of one mode in mode set 5 will result in the deletion of a rigid body mode since the mode set is an identity matrix. For this case consider a special definition of truncation by redefining the retained constraint modes for imposed displacement in the $\bar{B}$ set only. The retained set of modes will thus change as the size of the $\bar{B}$ set is reduced. This "special truncation" results in all $\bar{V}$ set physical coordinates being redefined as dependent degrees of freedom subject to the equations of constraint $F_v=0$. This "special truncation" is thus equivalent to the Guyan reduction‡ of Ref. 6. The truncated mode set results in no approximation with respect to statically applied forces in the $\bar{B}$ set unless inertial coordinates are truncated from an unrestrained model. For this case, the inertia relief forces in the $\bar{V}$ set are not zero and approximations or errors result.

Truncation of mode set 1 results in the deletion of unrestrained normal modes from the representation of the finite element model. The deflection pattern from the truncated mode set $x_\rho \equiv 0$ for static solutions if the retained mode set is statically complete. Therefore the equations of constraint are

$$x_\rho = \phi^n_\rho [(\phi^n_\rho)^T k \phi^n_\rho]^{-1} (\phi^n_\rho)^T F \equiv 0$$

or

$$(\phi^n_\rho)^T k \phi^n_\rho [(\phi^n_\rho)^T k\ \phi^n_\rho]^{-1} (\phi^n_\rho)^T F = (\phi^n_\rho)^T F \equiv 0$$

Modal mass orthogonality conditions will satisfy $(\phi^n_\rho)^T F = 0$ if external force patterns are expressable as a linear combination of inertia load patterns in the retained mode set. That is, if λ denotes the retained normal mode set and $\mathfrak{F}$ denotes a vector of constants such that

$$F = m\ \phi^n_\lambda\ \mathfrak{F}$$

then

$$(\phi^n_\rho)^T F = (\phi^n_\rho)^T m\ \phi^n_\lambda\ \mathfrak{F} \equiv 0$$

results from the orthogonality of the generalized mass matrix. Thus, if a truncated normal mode set is to be statically complete, all external forces imposed on the set must be linear combinations of the inertia force patterns in the retained modes. The poor convergence often experienced in a modal displacement forced response analysis is testimony to the difficulty of satisfying this requirement.

Truncation of mode set 2 results in the same equations of constraint as with set 1. Unless external forces are expressible as linear combinations of retained inertia load patterns, the truncated set will yield at best approximate displacement patterns for static analysis.

It is apparent that mode set 3 constraint modes result in a Guyan reduction of the $\bar{P}$ set to a $\bar{B}$ set of independent degrees of freedom. This set is supplemented with a cantilever normal mode set for the $\bar{V}$ set. Truncation of the normal mode set results in approximate static displacement patterns as previously discussed with set 1 and 2. If the $\bar{B}$ set is defined as the set of interface degrees of freedom for a component model, mode set 3 is equivalent to the Hurty[1] component mode set and identical to the Craig and Bampton[2] component mode set. The Hurty component mode set may thus be viewed as a Guyan reduction of the component model to the interface degrees of freedom supplemented with a set of normal modes cantilevered at the interface degrees of freedom.

It has been established that the modal truncation of an unrestrained finite element model may be alternately viewed as 1) truncation of normal modes, 2) truncation of inertia relief solutions, 3) a Guyan reduction, or 4) a combination of normal mode truncation plus a Guyan reduction. It follows that modal truncation in a transformed space is, in general, a combination of all four. It might be suspected that any truncated mode set representing an unrestrained finite element model may be most optimumly assembled using a combination of the above. A procedure is now developed to define a truncated modal representation of an unrestrained finite-element using such combinations.

‡Consider the constraint mode set of Eq. (1b). It yields

$$\begin{Bmatrix} x_b \\ \hline x_v \end{Bmatrix} = \begin{bmatrix} I \\ \hline -k_{vv}^{-1} k_{vb} \end{bmatrix} q^C = \begin{bmatrix} I \\ \hline -k_{vv}^{-1} k_{vb} \end{bmatrix} x_b$$

which is the definition of the transformation effected by a Guyan reduction in Ref. 6.

III. Definition of Truncated Component Mode Model

If economies are to be gained from a modal synthesis, it is obvious that component mode sets must be truncated. Degrees of freedom that are truncated from the component mode set are constrained out of the synthesized system. Any inadvertent truncation of active static degrees of freedom can only be compensated for by linear combinations of additional normal modes (usually inefficient). The objective here is to establish a mode set that will allow maximum truncation of normal modes. The philosophy of a modal acceleration forced response analysis is used to facilitate this objective. That is, the analysis is separated into two parts, statics§ and dynamics. A complete set of static interface modes is defined such that truncation of normal modes will delete at most the dynamic response of the truncated set. The complete interface mode set will retain the detail of the original finite element model with respect to statically imposed interface forces and displacements. Dynamic or normal modes may then be truncated using a frequency criterion.

A. Interface Mode Set

The most general component mode set with respect to interface statics is the unrestrained component with redundant interfaces. All other component models are a simplification of the unrestrained component. This analysis is for the static response of the unrestrained component with redundant interfaces. The $\bar{C}$ set of interface degrees of freedom is defined as: 1) $\bar{C}$—set of *all* interface degrees of freedom at *all* interfaces in a component model. Size of this set is $\bar{c}$, $\bar{p}$is generally much greater than $\bar{c}$. 2) $\bar{F}$—complement of $\bar{C}$ inthe component $\bar{P}$ set. The size of this set is $\bar{F} = \bar{p} - \bar{c}$.

The truncation of elastic degrees of freedom imposes equations of constraint upon the response of a finite-element model. It has been shown that these equations of constraint may be stated in terms of the forces applied to the model. Definition of constraint modes or attachment modes results in all finite element model coordinates with imposed zero force in Eqs. (1, 2, and 9) being redefined as dependent elastic degrees of freedom. There are a total of $\bar{p} - \bar{r}$ possible force equations of constraint in a finite-element model. The other $\bar{r}$ equations of constraint are displacement equations or rigid body modes. If the null set of forces is precisely stated there will be no approximation with respect to statics in a truncated set of interface modes. Two methods of determining such interface mode sets are defined here as the method of attachment modes and the method of constraint modes.

The method of attachment modes is defined for the unrestrained structure by $\bar{C}$ attachment modes for $\bar{L} = \bar{C}$ plus $\bar{r}$ rigid body modes. The attachment mode set is determined from Eqs. (9) which result in inertia relief modes. Equation 1 with $\bar{B} = \bar{R}$ (for any $\bar{R}$ set in $\bar{P}$) may be used to determine the required rigid body modes. The attachment modes will contain $\bar{r}$ independent inertia relief force patterns plus $\bar{c} - \bar{r}$ redundant elastic interface degrees of freedom.

The method of constraint modes is defined for the unrestrained structure using a combination of constraint and attachment modes. Equations (1) are used with $\bar{B} = \bar{C}$ to define $\bar{c}$ constraint modes. The equations of constraint in this set are $F_f = 0$. However, the uniformly accelerating

§Note that statics as used here includes the D'Alembert inertia forces resulting from rigid body acceleration response of the unrestrained component.

unrestrained component contains D'Alembert inertia reactions in the $\bar{F}$ set degrees of freedom. These reactions result from rigid body response to imposed interface forces and must therefore be accounted for in the statically complete interface mode set. These $\bar{F}$ set inertia reactions contain $\bar{r}$ independent [rank of $(\phi^r)^T m\phi^r$ in Eq. (8)] inertia relief force patterns. They may be most conveniently defined by Eq. (10) which retains zero displacements

$$\begin{bmatrix} m_{ff} & m_{fc} \\ m_{cf} & m_{cc} \end{bmatrix} \begin{bmatrix} \phi_f^r \\ \phi_c^r \end{bmatrix} + \begin{bmatrix} k_{ff} & k_{fc} \\ k_{cf} & k_{cc} \end{bmatrix} \begin{bmatrix} \phi_f^a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ F_c \end{bmatrix} \tag{10a}$$

$$\phi^a = \begin{bmatrix} -\phi_f^a \\ 0 \end{bmatrix} = \begin{bmatrix} k_{ff}^{-1}(m_{ff}\phi_f^r + m_{fc}\phi_c^r) \\ 0 \end{bmatrix} \tag{10b}$$

on the interface $\bar{C}$ set. Equations (9) may also be used by imposing unit forces on any $\bar{L}=\bar{R}$ set contained in $\bar{C}$, however, the $\bar{C}$ set will no longer contain zero displacement. *It is to be emphasized that the method of attachment modes and the method of constraint modes are equivalent.* Each results in a set of $\bar{c}+\bar{r}$ interface modes which is statically complete.

It is interesting to note simplifications in the method of attachment modes and the method of constraint modes when problem scope is reduced to a single nonredundant interface. For this case, the two methods yield an identical mode set; namely a set consisting of $\bar{r}$ attachment modes and $\bar{r}$ rigid body modes. Further, it is interesting to note the simplifications in the two methods if they are applied to an externally restrained component with redundant interface(s). The attachment modes will be defined by Eq. (2). The set of attachment modes and the set of constraint modes for this component will be related by a nonsingular, $\bar{c}\times\bar{c}$ linear transformation given by Eq. (4).

B. Dynamic Analysis for System Normal Modes

The interface mode set may be used with a supplementary set of component truncated normal modes to determine system normal modes with the synthesis equations in this paper. The higher-frequency component normal modes are truncated using a frequency criterion. The boundary conditions on the interface degrees of freedom for these modes may be fixed or free or any combination thereof. It is believed that free boundary conditions will yield more uniform convergence rates.

Care should be exercised if, for some reason, an untruncated component normal mode set is used. A normal mode set with free interface boundaries will contain most elastic plus all inertia relief degrees of freedom in the interface mode set resulting in up to $\bar{c}$ singularities in the component generalized coordinate space. An untruncated normal mode set with fixed interface boundaries will contain $\bar{r}$ singularities corresponding to the $\bar{r}$ inertia relief force patterns contained in the interface mode set. The inertia relief force patterns are implicitly contained in any untruncated component coordinate space.

C. Analysis for System Static Response

The interface mode set may be used with a supplementary set of static modes to perform static substructure analysis with the synthesis equations in this paper. The additional set of static modes required is defined for the $\bar{D}$ set: 1) $\bar{D}$—set of all degrees of freedom to which external forces are to be applied in the component structure $\bar{F}$ set. The size of this set is $\bar{d}$. 2) $\bar{E}$—complement of $\bar{D}$ in $\bar{F}$.

The supplementary static mode set releases the equations of constraint $F_d=0$. It is most straightforwardly defined by attachment modes or constraint modes with zero displacement imposed on $\bar{C}$, the interface degrees of freedom. The resulting response in the component $\bar{P}$ set, of course, includes the imposed zeros in $\bar{C}$. The nonzero deflections in these mode sets are obtained from Eq. (1b) with $\bar{B}=\bar{D}$ and $\bar{V}=\bar{E}$, or Eq.(2b) with $\bar{L}=\bar{D}$ and $\bar{W}=\bar{E}$. Either the attachment modes or the constraint modes may be used, but not both, as one is a linear combination of the other. The total number of component modes required for static analysis is thus $\bar{r}+\bar{c}+\bar{d}$ for each component. If the $\bar{D}$ set is untruncated, the static mode set will contain $\bar{r}$ redundant inertia relief force patterns.

IV. Structural Synthesis

Modal synthesis is directly analogous to the assembling together of component mass and stiffness matrices for structural components. However, modal synthesis is somewhat more complicated since the coupling coordinates in these analyses are dependent physical coordinates. The independent coordinates are, of course, the component modal degrees of freedom of the various components. When component mass and stiffness matrices are assembled together, the coupling coordinates are independent coordinates. Hence the mathematics (or at least the procedure) for direct assembling of mass and stiffness matrices is much simpler than for synthesis.

The synthesis technique proposed is a general coupling procedure which requires the following characteristics in a component mode shape matrix. First, the modes in the mode shape matrix must be linearly independent if the modal coordinates are to be used in subsequent dynamic analysis. Second, if $\bar{c}$ physical coordinates of the mode shape matrix are to be coupled with another component, $\bar{c}$ interface coordinates and $\bar{c}$ columns must be partitionable into a square, $\bar{c}\times\bar{c}$, nonsingular partition [see Eq. (14)]. For the method of constraint modes, this nonsingular partition is a $\bar{c}\times\bar{c}$ identity matrix [see Eq. (1a)]. For the method of attachment modes this matrix is defined by $\phi_l^a(\bar{L}$ set $=\bar{C}$ set) in Eq. (2a) fora restrained component and ϕ_f^a in Eq. (9) for an unrestrained component.

This analysis is formulated for a two-component synthesis but can be readily extended to more components. The coupling procedure is similar to that found in Ref. 5, which is based upon the assumptions that external forces are zero and interface component displacements are expressed in the same coordinate system. In general, external forces are nonzero for static analysis; and component coordinate systems, especially those from different organizations, will differ. Hence, the coupling procedure is rederived here to include these features.

The equation of motion for an undamped structure in a set of system coordinates is:

$$m\ddot{x}+kx=F \tag{11}$$

Consider the structure to be made up of two discrete components. The equation of motion for the structure in terms of its uncoupled component coordinates x_i, x_j interface reaction forces R_i, R_j and external forces F_i, F_j can be written as

$$\begin{bmatrix} m_i & 0 \\ 0 & m_j \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_j \end{Bmatrix} + \begin{bmatrix} k_i & 0 \\ 0 & k_j \end{bmatrix} \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{Bmatrix} R_i \\ R_j \end{Bmatrix} + \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{12}$$

The component coordinates may be defined by a set of component modes as

$$\begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{bmatrix} \phi_i & 0 \\ 0 & \phi_j \end{bmatrix} \begin{Bmatrix} p_i \\ p_j \end{Bmatrix} \tag{13}$$

The component mode shape matrices may be partitioned to separate the set of interface coordinates from remaining physical coordinates.

$$\phi = \left[\begin{array}{c|c} \phi^{ff} & \phi^{fc} \\ \hline \phi^{cf} & \phi^{cc} \end{array}\right] \bar{c} \text{ rows} \qquad (14)$$

$\bar{c}$ columns

where ϕ^{cc} is the matrix partition of ϕ such that rows consist of interface physical coordinates in $\bar{C}$ set. Columns are selected such that ϕ^{cc} is nonsingular. The remainder of component modes are ϕ^{ff}, ϕ^{fc}, ϕ^{cf}, as shown in Eq. (14).

The equation of motion in uncoupled component modal coordinates may be written as

$$\left[\begin{array}{c|c} \phi_i^T m_i \phi_i & 0 \\ \hline 0 & \phi_j^T m_j \phi_j \end{array}\right] \left\{\begin{array}{c} \ddot{p}_i \\ \hline \ddot{p}_j \end{array}\right\} + \left[\begin{array}{c|c} \phi_i^T k_i \phi_i & 0 \\ \hline 0 & \phi_j^T k_j \phi_j \end{array}\right] \left\{\begin{array}{c} p_i \\ \hline p_j \end{array}\right\}$$

$$= \left\{\begin{array}{c} \phi_i^T R_i \\ \hline \phi_j^T R_j \end{array}\right\} + \left\{\begin{array}{c} \phi_i^T F_i \\ \hline \phi_j^T F_j \end{array}\right\} \qquad (15)$$

$$\left[\begin{array}{c|c} \mathfrak{M}_i & 0 \\ \hline 0 & \mathfrak{M}_j \end{array}\right] \left\{\begin{array}{c} \ddot{p}_i \\ \hline \ddot{p}_j \end{array}\right\} + \left[\begin{array}{c|c} \kappa_i & 0 \\ \hline 0 & \kappa_j \end{array}\right] \left\{\begin{array}{c} p_i \\ \hline p_j \end{array}\right\}$$

$$= \left\{\begin{array}{c} \phi_i^T R_i \\ \hline \phi_j^T R_j \end{array}\right\} + \left\{\begin{array}{c} \phi_i^T F_i \\ \hline \phi_j^T F_j \end{array}\right\} \qquad (16)$$

where

$$\mathfrak{M}_i = \phi_i^T m_i \phi_i \quad ; \quad \mathfrak{M}_j = \phi_j^T m_j \phi_j$$

$$\kappa_i = \phi_i^T k_i \phi_i \quad ; \quad \kappa_j = \phi_j^T k_j \phi_j \qquad (17)$$

The equations of constraint for the connected components are

$$x_j^c = S x_i^c \qquad (18a)$$

$$R_j^c + S R_i^c = 0 \qquad (18b)$$

where S is the unitary coordinate transformation from the coordinate system of component i to that of component j for interface coordinates. But

$$x_i^c = \left[\begin{array}{c|c} \phi_i^{cf} & \phi_i^{cc} \end{array}\right] \left\{\begin{array}{c} p_i^f \\ \hline p_i^c \end{array}\right\} \qquad (19)$$

$$x_j^c = \left[\begin{array}{c|c} \phi_j^{cf} & \phi_j^{cc} \end{array}\right] \left\{\begin{array}{c} p_j^f \\ p_j^c \end{array}\right\} \qquad (20)$$

Therefore, Eq. (18a) becomes

$$\phi_j^{cf} p_j^f + \phi_j^{cc} p_j^c = S \left[\begin{array}{c|c} \phi_i^{cf} & \phi_i^{cc} \end{array}\right] \left\{\begin{array}{c} p_i^f \\ p_i^c \end{array}\right\} \qquad (21)$$

and

$$p_j^c = \left[\phi_j^{cc}\right]^{-1} \left[S \left[\begin{array}{c|c} \phi_i^{cf} & \phi_i^{cc} \end{array}\right] \left\{\begin{array}{c} p_i^f \\ p_i^c \end{array}\right\} - \phi_j^{cf} p_j^f \right] \qquad (22)$$

thus

$$\left\{\begin{array}{c} p_i^f \\ \hline p_i^c \\ \hline p_j^f \\ \hline p_j^c \end{array}\right\} = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \\ \hline [\phi_j^{cc}]^{-1} S \phi_i^{cf} & [\phi_j^{cc}]^{-1} S \phi_i^{cc} & -[\phi_j^{cc}]^{-1} \phi_j^{cf} \end{array}\right] \left\{\begin{array}{c} p_i^f \\ \hline p_i^c \\ \hline p_j^f \end{array}\right\} \qquad (23)$$

Note that the first 3 rows of partitions in Eq. (23) are identities and the bottom partition is Eq. (22). p_j^c have been made dependent degrees of freedom in the coupled system. Let

$$y = \left\{\begin{array}{c} p_i^f \\ \hline p_i^c \\ \hline p_j^f \end{array}\right\} \qquad (24)$$

$$\beta = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \\ \hline [\phi_j^{cc}]^{-1} S \phi_i^{cf} & [\phi_j^{cc}]^{-1} S \phi_i^{cc} & -[\phi_j^{cc}]^{-1} \phi_j^{cf} \end{array}\right] \qquad (25)$$

$$\left\{\begin{array}{c} p_i^f \\ \hline p_i^c \\ \hline p_j^f \\ \hline p_j^c \end{array}\right\} = \left\{\begin{array}{c} p_i \\ \hline p_j \end{array}\right\} = \beta y \qquad (26)$$

Substituting Eqs. (18b) and (26) into Eq. (16), and premultiplying by β^T yields Eq. (27), the equation of motion in coupled component coordinates.

$$\beta^T \left[\begin{array}{c|c} \mathfrak{M}_i & 0 \\ \hline 0 & \mathfrak{M}_j \end{array}\right] \beta \ddot{y} + \beta^T \left[\begin{array}{c|c} \kappa_i & 0 \\ \hline 0 & \kappa_j \end{array}\right] \beta y$$

$$= \beta^T \left\{\begin{array}{c} \phi_i^T \; R_i \\ \hline -\phi_j^T \; S R_i \end{array}\right\} + \beta^T \left\{\begin{array}{c} \phi_i^T \; F_i \\ \hline \phi_j^T \; F_j \end{array}\right\} \qquad (27)$$

since the constraint reactions in partitioned form are

$$\left\{\begin{array}{c} 0 \\ \hline R^c \end{array}\right\}$$

$$\phi_i^T R_i = \left[\begin{array}{c|c} (\phi_i^{ff})^T & (\phi_i^{cf})^T \\ \hline (\phi_i^{fc})^T & (\phi_i^{cc})^T \end{array}\right] \left\{\begin{array}{c} 0 \\ \hline R_i^c \end{array}\right\}$$

$$= \left\{\begin{array}{c} (\phi_i^{cf})^T R_i^c \\ \hline (\phi_i^{cc})^T R_i^c \end{array}\right\} \qquad (28)$$

Similarly

$$\phi_j^T S R_i = \left\{\begin{array}{c} (\phi_j^{cf})^T S R_i^c \\ \hline (\phi_j^{cc})^T S R_i^c \end{array}\right\} \qquad (29)$$

Combining Eqs. (25, 28, and 29) the reaction forces of Eq. (27) become

$$\beta^T \left\{ \begin{array}{c} \phi_i^T R_i \\ \hline -\phi_j^T S R_i \end{array}\right\} = \left[\begin{array}{c|c|c|c} I & 0 & 0 & (\phi_i^{cf}) S^T [(\phi_j^{cc})^T]^{-1} \\ \hline 0 & I & 0 & (\phi_i^{cc})^T S^T [(\phi_j^{cc})^T]^{-1} \\ \hline 0 & 0 & I & -(\phi_j^{cf})^T [(\phi_j^{cc})^T]^{-1} \end{array}\right] \left\{\begin{array}{c} (\phi_i^{cf})^T R_i^c \\ \hline (\phi_i^{cc})^T R_i^c \\ \hline -(\phi_j^{cf})^T S R_i^c \\ \hline -(\phi_j^{cc})^T S R_i^c \end{array}\right\}$$

$$= \left\{\begin{array}{c} (\phi_i^{cf})^T R_i^c - (\phi_i^{cf})^T S^T [(\phi_j^{cc})^T]^{-1} (\phi_j^{cc})^T S R_i^c \\ \hline (\phi_i^{cc})^T R_i^c - (\phi_i^{cc})^T S^T [(\phi_j^{cc})^T]^{-1} (\phi_j^{cc})^T S R_i^c \\ \hline -(\phi_j^{cf})^T S R_i^c + (\phi_j^{cf})^T [(\phi_j^{cc})^T]^{-1} (\phi_j^{cc})^T S R_i^c \end{array}\right\} \equiv \left\{\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \end{array}\right\}$$

since S is unitary and $S^T = S^{-1}$. The coupled equation of motion for the system [Eq. (27)] therefore becomes

$$\beta^T \left[\begin{array}{c|c} \mathfrak{M}_i & 0 \\ \hline 0 & \mathfrak{M}_j \end{array}\right] \beta \ddot{y} + \beta^T \left[\begin{array}{c|c} \kappa_i & 0 \\ \hline 0 & \kappa_j \end{array}\right] \beta y$$

$$= \beta^T \left\{\begin{array}{c} \phi_i^T F_i \\ \hline \phi_j^T F_j \end{array}\right\} \tag{30}$$

Equations (13, 26, and 30) may be used to solve for system static response to external forces applied to component $\bar{c}$ and $\bar{D}$ set degrees of freedom. Techniques similar to those used in deriving Eq. (9) are required to solve the unrestrained uniformly accelerating system. Dynamic analysis for system normal modes results in no external forces on the system, i.e., $F_i = F_j = 0$. Equation (30) thus results in the classical eigenvalue problem which may be solved for eigenvalues and eigenvector matrix ξ. The normal mode shape matrix for the coupled system is then given by

$$\phi^s = \left[\begin{array}{c|c} \phi_i & 0 \\ \hline 0 & \phi_j \end{array}\right] \beta \xi \tag{31}$$

V. Numerical Results

Application of the mode sets previously discussed is demonstrated by solving a planar truss problem found in Ref. 7. This truss problem consists of a two-component synthesis for the truss components which are given in Fig. 1. The interface contains 6 degrees of freedom; each component has 3 rigid body modes. There are 3 redundant elastic degrees of freedom at the interface and furthermore there are 3 inertia relief solutions for each component.

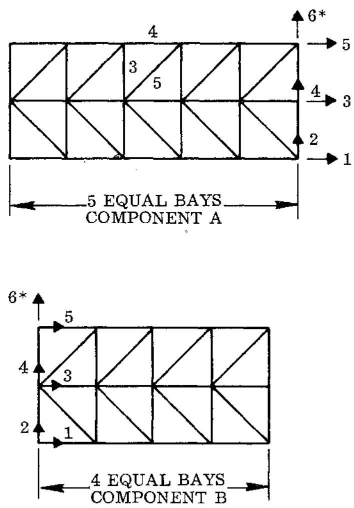

Fig. 1 Truss model. All members have constant area and uniform mass properties. Each joint has two in-plane degrees of freedom resulting in 60 degrees of freedom for the complete truss. * signifies interface degrees of freedom.

The truss problem is solved using four methods; 1) the method of attachment modes with unrestrained normal modes, 2) the method of attachment modes with cantilevered

Table 2 Percent frequency error using 12 degrees of freedom

Elastic mode number	Unrestrained normal + method of attachment modes	Cantilever normal + method of attachment modes	Cantilever normal + method of constraint modes	Hurty
1	0.00592	0.000669	0.000669	0.0112
2	0.0193	0.187	0.187	0.0128
3	0.0743[a]	0.743[a]	0.743[a]	0.0313
4	0.150	2.94	2.94	0.155
5	1.68	10.3	10.3	0.190
6	6.55	16.9	16.9	0.184
7	16.8	16.9	16.9	7.39
8	11.31	23.8	23.8	4.88
9	22.1	52.4	52.4	22.6[a]
10	48.9	63.6	63.6	47.9
11	102.0	102.0	102.0	102.0
12	181.0	170.0	170.0	175.0

[a] Last dynamic DOF.

Table 3 Percent frequency error using 20 elastic degrees of freedom

Elastic mode number	Unrestrained normal + method attachment modes	Cantilever normal + method of attachment modes	Cantilever normal + method of constraint modes	Hurty
1	0.0000171	0.00000000910	0.00000000934	0.000743
2	0.0000606	0.00000335	0.00000335	0.00177
3	0.0138	0.00584	0.00584	0.00962
4	0.000238	0.0000188	0.0000188	0.00921
5	0.000806	0.00139	0.00139	0.0335
6	0.00197	0.000539	0.000539	0.0103
7	0.0832	0.264	0.264	0.941
8	0.00679	0.0178	0.0178	0.117
9	0.000932	0.690	0.690	0.797
10	0.00447	0.249	0.249	0.197
11	0.0216[a]	1.03[a]	1.03[a]	0.302
12	0.134	11.1	11.1	0.279
13	5.33	10.0	10.0	0.138
14	7.15	18.5	18.5	0.718
15	15.9	34.0	34.0	2.63
16	29.2	36.2	36.2	11.4
17	24.1	50.9	50.9	10.6[a]
18	44.8	56.6	56.6	11.2
19	47.0	65.1	65.1	35.7
20	92.1	93.3	93.3	90.6

[a] Last dynamic DOF.

Table 4 Percent frequency error using 28 elastic degrees of freedom

Elastic mode number	Unrestrained normal + method of attachment modes	Cantilver normal + method of attachment modes	Cantilever normal + method of constraint modes	Hurty
1	0.0000121	0.00000000110	0.00000000138	0.000276
2	0.0000347	0.00000000662	0.00000000702	0.000434
3	0.00120	0.00279	0.00279	0.00642
4	0.000138	0.00000119	0.00000119	0.00319
5	0.000437	0.00000300	0.00000301	0.00687
6	0.000844	0.00000882	0.00000882	0.00293
7	0.00140	0.00454	0.00454	0.353
8	0.00249	0.000152	0.000152	0.0278
9	0.0000444	0.00103	0.00103	0.147
10	0.000348	0.000155	0.000155	0.0234
11	0.00443	0.000699	0.000699	0.0511
12	0.0161	0.000713	0.000714	0.0403
13	0.0669	0.0702	0.0702	0.0945
14	0.00212	0.134	0.134	0.250
15	0.0175	1.26	1.26	1.18
16	0.00867	0.0342	0.0342	0.564
17	0.105	1.04	1.04	0.335
18	1.03	0.669	0.669	0.262
19	1.16[a]	6.30[a]	6.30[a]	1.41
20	2.44	3.81	3.81	0.834
21	6.78	9.19	9.19	0.545
22	15.0	20.0	20.0	0.539
23	22.2	22.8	22.8	0.745
24	22.9	40.7	40.7	2.88
25	25.7	31.6	31.6	5.24[a]
26	31.6	38.4	38.4	10.2
27	37.3	37.9	37.9	14.1
28	62.9	63.2	63.2	61.3

[a] Last dynamic DOF.

normal modes (the cantilevers are imposed at the interface degrees of freedom of the two components), 3) the method of constraint modes with the cantilever normal modes, and 4) Hurty's method. Hurty's method is method 3 without inertia relief solutions. Results are presented in Tables 2-5 in the form of resultant frequency error for a synthesis using 12, 20, 28, and 36 component elastic degrees of freedom, respectively. In all cases a frequency cutoff criterion is used to truncate higher frequency component normal modes.

All three solutions using the method of attachment modes or the method of constraint modes contain nine elastic steady state or static modes; Hurty's method contains three such

Table 5 Percent frequency error using 36 elastic degrees of freedom

Elastic mode number	Unrestrained normal + method of attachment modes	Cantilever normal + method of attachment modes	Cantilever normal + method of constraint modes	Hurty
1	0.00000319	0.000000000141	0.000000000402	0.000269
2	0.00000658	0.00000000503	0.00000000562	0.000151
3	0.000186	0.000774	0.000774	0.00102
4	0.0000420	0.0000000140	0.0000000136	0.00304
5	0.0000955	0.00000133	0.00000135	0.00199
6	0.000254	0.000000242	0.000000242	0.00249
7	0.000274	0.000231	0.000231	0.122
8	0.000489	0.00000630	0.00000631	0.0211
9	0.0000309	0.000141	0.000142	0.0585
10	0.0000188	0.0000265	0.0000267	0.00775
11	0.000832	0.0000382	0.0000383	0.0327
12	0.00301	0.0000730	0.0000732	0.0175
13	0.00754	0.0180	0.0180	0.0101
14	0.000160	0.104	0.104	0.107
15	0.00103	0.228	0.228	0.0529
16	0.00218	0.00246	0.0027	0.249
17	0.0101	0.0885	0.0885	0.156
18	0.0123	0.00493	0.00493	0.0791
19	0.0178	0.230	0.230	0.0725
20	0.0113	0.0321	0.0321	0.0340
21	0.114	0.0111	0.0111	0.208
22	0.0169	0.214	0.214	0.0814
23	0.0240	0.00842	0.00842	0.00681
24	0.0516	0.104	0.104	0.104
25	0.474	0.967	0.967	0.202
26	0.0981	2.79	2.79	0.408
27	1.24[a]	0.377[a]	0.377[a]	0.390
28	1.20	2.60	2.60	0.777
29	4.09	3.81	3.81	0.545
30	5.55	11.2	11.2	0.322
31	7.93	11.5	11.5	0.318
32	7.70	16.6	16.6	1.28
33	13.3	15.5	15.5	2.12[a]
34	15.5	17.2	17.2	2.57
35	18.1	18.7	18.7	3.31
36	34.6	34.6	34.6	33.0

[a] Last dynamic DOF.

modes. The number of component normal modes that were used in the method of attachment modes and the method of constraint modes in Tables 2–5 is, respectively, 3, 11, 19, and 27 modes; Hurty's method contains 9, 17, 25, and 33 component normal modes.

Thus, in every example, the new methods contain six less component normal modes than the existing method. And yet, the new methods yield significant overall improvement in accuracy when compared to the existing method, except for the last nine frequency errors. For the last nine modes, the existing method yields a better frequency error than the new methods; however, the error is still considerable.

It is interesting to note that in all tables where the method of attachment modes and the method of constraint modes are used with a cantilever normal mode set, the frequency errors are identical except where they are so small as to be calculated zeros. This demonstrates the equivalence of the method of attachment modes and the method of constraint modes.

It may also be noted that the solution using the method of attachment modes and an unrestrained set of normal modes appears to be more uniformly accurate than that using the set of cantilever normal modes. That is, there are no modes with frequency errors of 0.1 or 0.2% in the lower modes. The cantilever modes, on the other hand, yield some modes with uncanny accuracy in frequency; accuracies of 10 or 12 places. However, the corresponding frequency calculations for unrestrained mode set are still accurate to some 8 or 9 places. It would seem that using the method that gives significant overall accuracy is worth the sacrifice of a few more closely accurate frequencies obtained from a cantilevered mode set.

When a synthesis is performed with a dynamic normal mode set at an unrestrained boundary, it is apparent that partial constraints are imposed on those dynamic modes. If the synthesis is performed with respect to a cantilevered boundary, a partial release of this boundary occurs. From a philosophical point of view, it seems that the more suitable method would be to partially constrain dynamic modes in a synthesis rather than to partially release dynamic modes in a synthesis. In any event, the use of an unrestrained boundary in the dynamic normal mode set seems to give a better overall representation of the component structure in the system synthesis.

VI. Conclusions

All modal truncation techniques considered in this paper impose equations of constraint upon component external forces. Displacement response of any truncated set will be approximate with respect to statics unless external forces applied to the mode set satisfy the imposed equations of constraint. The use of constraint and attachment modes imposes equations of constraint of the form $F=0$ on some subset of a finite element model physical coordinate space. The equations of constraint for a normal mode set are twofold. First, $F=0$

for all unconstrained physical coordinates with zero mass or inertia. Second, external forces must satisfy the equations of constraint $(\phi_p^n)^T F=0$ for all truncated normal mode sets.

Modal acceleration techniques in forced response analysis with normal modes avoid these constraints with separate static solutions. In component modal synthesis, the static solution must be included in the component normal modes or the interface mode set. The equations of constraint on a normal mode set are unwieldy and difficult to deal with when compared to the equations of constraint on attachment and constraint modes. It is therefore attractive to express static response in the form of the more easily manipulated interface mode set.

Previous modal synthesis techniques have imposed cantilever boundary conditions[1,2] on component normal modes or used interface inertia and/or stiffness loading.[7] All such techniques approximate component static response to interface loading in the component normal mode set. The interface mode sets proposed in this paper include all component static response to interface loading in the interface mode set. The interface mode set is statically complete. Interface loading or constraints are no longer required on component normal modes when used with a statically complete interface mode set and interface boundary conditions on component normal modes may thus be fixed or free. The statically complete interface mode set will generally be a linear combination of an untruncated normal mode set. However, any normal mode set with appreciable truncation will cause no problem in this regard.

The two new interface mode sets recommended in this paper, namely the method of constraint modes and the method of attachment modes,¶ are statically complete and are therefore equivalent. This equivalence is demonstrated in Tables 2–5 wherein both methods are used with identical truncated cantilever normal mode sets and yield precisely the same results (except for some calculated zeros). The synthesis equations of this paper may be used with either method to perform static or dynamic analysis. The interface mode set is supplemented with component normal modes for dynamic analysis. It is supplemented with additional static modes for static analysis. Bamford's attachment modes,[3] Hurty's constraint modes[1] and the Guyan reduction[6] all yield an identical reduction of a restrained finite element model. This is to be expected, as all three techniques impose precisely the same equations of constraint upon the model. That is, $F=0$ for all dependent degrees of freedom. The five desirable characteristics of modal synthesis analysis put forth in Sec. I are now examined.

¶Each of these methods actually uses a combination of constraint modes and attachment modes for the unrestrained component with rigid body degrees of freedom.

1) *Computational efficiency:* The methods of this paper incorporate all component static response to interface loading into the interface mode set; hence, only the dynamic response of normal modes is truncated from synthesis analysis. Higher frequency component normal modes may be truncated using a frequency criterion. It will not be necessary to carry any component normal modes to insure static convergence in stress calculations.

2) *Interchangeability:* The component interface mode sets defined in this paper are independent of adjacent component properties and may, therefore, be used in any structural system with compatible interfaces.

3) *Component flexibility:* The truncation of any one attachment mode, in the method of attachment modes, effects no change on the retained interface mode set. This method may, therefore, be used to define optional interface modes in component models which may be discarded at the time of synthesis.

4) *Synthesis flexibility:* The synthesis equation of this paper[5] are general coupling equations which may be used to couple either fixed or free interface component normal modes. They may be used with inertia loading techniques on nonredundant interfaces.

5) *Modal acceleration analysis:* Modal acceleration coefficients required for force response analysis of synthesized systems may be determined using the static analysis of this paper. Stresses due to the static application of external forces are readily available for all components in the synthesized system. Once component mode sets are determined for dynamic analysis, the number of additional modes required for static analysis is equal to or less than the number of external forces applied to the system.

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